ON THE STABILITY OF SPHERICALLY SYMMETRIC CONFIGURATIONS IN NEWTONIAN LIMIT OF JORDAN, BRANS-DICKE THEORY.

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Abstract

We discuss stability of spherically symmetric static solutions in Newtonian limit of Jordan, Brans-Dicke field equations. The behavior of the stable equilibrium solutions for the spherically symmetric configurations considered here, it emerges that the more compact a model is, the more stable it is. Moreover, linear stability analysis shows the existence of stable configurations for any polytropic index.

1   Introduction

Scalar-tensor theories, in which a long-range scalar field in addition to the usual tensor fields present in Einstein's theory mediates gravity, have been studied in many works (see e.g.[1],[2]) as natural generalizations of Einstein's general relativity (GR). The simplest of them Jordan, Brans-Dicke (JBD) theory of gravity [3],[4], in which a scalar field ϕ acts as the source for the gravitational coupling with $G \sim \frac{1}{\phi}$, was essentially motivated by apparent discrepancies between observations and the weak-field predictions of GR. It is a well-known fact that most of the mathematical difficulties of theories of gravity lie in the high non-linearity of the field equations. However, under the special circumstance when the gravitational field is weak one can linearize the field equations thereby ignoring this feedback effect. Then the weak-field limit is analyzed and the conditions leading to significant deviations of the $1/r^2$ Newton's law of gravitation are discussed.

The stability of equilibrium configurations for both stars and star clusters remains a topic of continuing interest. The fundamental dynamics describing galaxies, clusters of galaxies, or globular clusters [5], is a collisionless system. In the general case the collisionless Boltzmann equation cannot be solved because it involves too many independent variables. However, we can get certain system with a polytropic state equation, corresponding to isotropic velocity dispersion tensors. Moreover, spherically symmetric static Newtonian perfect fluid models are the starting point for many discussions about stellar structure and evolution. Within this class of models, polytropic equations of state have been studied thoroughly [6], [7], [8], [9], [10]

$$P = K\rho^\gamma \equiv K\rho^{1+\frac{1}{n}},$$

where K is non-negative polytropic constant, γ is the adiabatic index, and n the politropic index. In astrophysics pure polytropes have been considered and some of the corresponding stellar models have been studied in detail in many textbooks [11], although mostly numerically.

In this paper we consider static spherically symmetric perfect fluid models using Newtonian approximations of Jordan, Brans-Dicke (JBD) theory of gravity. This paper is organized as follows. In section 2 we have constructed the field equations for the Newtonian limit of JBD theory. In section 3 we derive the basic equation with approximation of small linear perturbation and give results on stability versus instability for spherically symmetric static models. The paper ends with a conclusion in section 4.

2   Basic equations and their properties

In this section we discuss the weak-field limit of Jordan, Brans-Dicke theory [3],[4]. The JBD theory incorporates the Math principle, which states that the phenomenon of inertia must arise from accelerations with respect to the general mass distribution of the universe. Differences between predictions of JBD theory and observing appeared at study of deciding the field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{c^4\phi}T_{\mu\nu} - \frac{\omega}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) - \frac{1}{\phi}\left(\phi_{,\mu;\nu} - g_{\mu\nu}\phi_{;\lambda}^{;\lambda}\right), \quad (1)$$

$$\phi_{;\lambda}^{;\lambda} = -\frac{8\pi}{c^4(3+2\omega)}T, \tag{2}$$

Equations of gravitation field allow significant simplification, if velocity of material point far less then velocity of light, so values of order v^2/c^2 possible neglect. In the case of weak field approximation value $g_{\mu\nu}$ must extremely little differ from:

$$\begin{aligned} g_{\mu\nu} &= 1 \quad for \ \mu = \nu = 1,2,3, g_{00} = -1 \\ g_{\mu\nu} &= 0 \quad for \ \mu \neq \nu \end{aligned}$$

and squares of these deflections possible to neglect. Then

$$\frac{\partial^2 x^\mu}{\partial t^2} = -c^2 \Gamma_{00}^\mu, \tag{3}$$

Moreover, in static case derived $g_{\mu\nu}$ on time possible to neglect too. Then one can change Γ_{00}^μ to $\Gamma_{\mu,00}$, or $-\frac{1}{2}\frac{dg_{00}}{dx^\mu}$ and equations of motion a material point (3) for small velocities and weak field takes Newtonian form:

$$\frac{\partial^2 x^\mu}{\partial t^2} = -\frac{\partial U}{\partial x^\mu}$$

where U is a gravitation potential, and

$$g_{00} = 1 - \frac{2U}{c^2}. \tag{4}$$

From equations (1), (2) one can get

$$R_{\mu\nu} + \frac{\omega}{\phi^2}\phi_{;\mu}\phi_{;\nu} + \frac{1}{\phi}\phi_{;\mu;\nu} = -\frac{8\pi}{c^4\phi}\left[T_{\mu\nu} - \frac{1+\omega}{3+2\omega}Tg_{\mu\nu}\right]. \tag{5}$$

For the component 00 equations (5) values of order v/c possible neglect, except T_{00}. Component of energy momentum tensor $T_{00} = \rho c^2$ consequently $T = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} = -\rho c^2$, and

$$R_{00} + \frac{1}{\phi}\phi_{;0;0} = -\frac{8\pi}{c^2\phi}\frac{2+\omega}{3+2\omega}\rho$$

As far as derived on time and product $\Gamma_{\nu\lambda}^{\mu}$ we neglect, then

$$R_{00} = \frac{\partial\Gamma_{00}^i}{\partial x}$$

since $\Gamma_{00}^i \approx \Gamma_{i,00} \approx -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i}$ then from (4)

$$R_{00} = \frac{1}{2}\sum_i \frac{\partial^2 g_{00}}{\partial x_i^2} = \frac{1}{2}\triangle g_{00} = -\frac{\triangle U}{c^2}.$$

For scalar potential we have $\phi_{;0;0} = 2\Gamma_{00}^i \frac{\partial\phi}{\partial x^i}$.
Finely from (1) and (2)

$$div\,(\phi\nabla U) = 8\pi\frac{2+\omega}{3+2\omega}\rho, \tag{6}$$

$$\triangle\phi = -\frac{8\pi}{3+2\omega}\rho. \tag{7}$$

where ρ is density of mater. Limiting transformation to Newton theory of gravitation occurs when $|\,\omega\,| \rightarrow \infty$ and $\phi = const.$

3   Dynamical stability

We discuss the problem of dynamical stability of the equilibrium solutions and consider small time dependent radial perturbations, which still preserve spherical symmetry. Let's consider oscillations of configurations consisting of ideal gas with adiabatic radial perturbations. Generally the solutions of equations of stars oscillations are connected with big mathematical difficulties, therefore we assume the approximation of small perturbation [12]. Let's express w, u and v as radial, latitude and meridianal components of speed in spherical coordinates:

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$$X = div\left(\frac{\partial \xi}{\partial t}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w\right) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \theta}\left(u\,\sin\theta\right) + \frac{1}{r\,\sin\theta}\frac{\partial v}{\partial \varphi}, \qquad (8)$$

then from the equation of continuity

$$\frac{d}{dt}\left(\rho_0 + \rho\right) = -\left(\rho_0 + \rho\right)X, \qquad (9)$$

we have:

$$\frac{\partial \rho}{\partial t} = -w\frac{\partial \rho_0}{\partial r} - \rho_0 X, \qquad (10)$$

Throughout the paper, we use values with zero designate not perturbed values, and without zero a perturbation of the value. In case of ideal gas the adiabatic condition will be written down so

$$\frac{1}{P_0 + P}\frac{d}{dt}\left(P_0 + P\right) = \frac{\gamma}{\rho_0 + \rho}\frac{d}{dt}\left(\rho_0 + \rho\right). \qquad (11)$$

In galactic dynamics γ is less than 3 [5] which means that no polytropic stellar system can be homogeneous. In real stars γ is the variable and ranges from 1 to ∞, but possible changes of it are rather small therefore we assume for simplicity, that it is a constant. We use for convenience, values ε and g determined as:

$$\begin{aligned}\varepsilon^2 &= \gamma\frac{P_0}{\rho_0}, \\ g &= \nabla U.\end{aligned}$$

Then from an adiabatic condition we find

$$\frac{\partial P}{\partial t} + w\frac{\partial P_0}{\partial r} = \varepsilon^2\left(\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_0}{\partial r}\right) = -\rho_0\varepsilon^2 X. \qquad (12)$$

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From

$$\frac{\partial P_0}{\partial r} = -g_0 \rho_0$$

and (12) we obtain

$$\frac{\partial P}{\partial t} = \rho_0 \left(g_0 w - \varepsilon^2 X \right). \tag{13}$$

The equations for gravitational potential satisfy

$$div \left(\phi_0 g_0 \right) = 8\pi \frac{2 + \omega}{3 + 2\omega} \rho_0$$

Perturbation of a gravitational field is given by

$$div \left(\phi g_0 + \phi_0 g \right) = 8\pi \frac{2 + \omega}{3 + 2\omega} \rho, \tag{14}$$

The gravitational scalar consist the equilibrium and perturbed part which satisfy to the equations

$$div \left(\nabla \phi_0 \right) = -\frac{8\pi \rho_0}{3 + 2\omega}, \tag{15}$$

$$div \left(\nabla \phi \right) = -\frac{8\pi \rho}{3 + 2\omega}. \tag{16}$$

The equations governing the linear perturbations are obtained by expanding all functions to first order:

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial \theta} \left(P + \rho_0 \varphi \right), \tag{17}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left(P + \rho_0 \varphi \right), \tag{18}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial P}{\partial r} - g_0 \rho - \rho_0 g. \tag{19}$$

Furthermore, we suppose a time dependence of the form

$$A(r,t) = A(r) e^{i\sigma t}$$

We differentiate (19) on t and (13) on r then taking into account the equation of continuity it is received to

$$\sigma^2 w - g_0' w - g_0 w' - i\sigma g + \varepsilon^2 X' + g_0 (1 - \gamma) X = 0, \tag{20}$$

where the prime means derivative with respect to r. In a case of radial pulsations (8) it transformed to

$$X = w' + \frac{2w}{r}, \tag{21}$$

The equations (14) and (16) transforms to

$$i\sigma g = -\frac{8\pi (2 + \omega) \rho_0 w}{(3 + 2\omega) \phi_0} - \frac{i\sigma g_0 \phi}{\phi_0}, \tag{22}$$

$$i\sigma \frac{\partial \phi}{\partial r} = \frac{8\pi \rho_0}{3 + 2\omega} w. \tag{23}$$

Then the equation (20) takes a form

$$
\begin{aligned}
-\sigma^2 w &= -g_0 w' + \frac{i\sigma g_0 \phi}{\phi_0} + \varepsilon^2 \left(w'' + \frac{2w'}{r} - \frac{2w}{r^2} \right) + \\
&\quad + g_0 (1 - \gamma) \left(w' + \frac{2w}{r} \right) + \frac{2}{r} w g_0 + \frac{g_0 \phi_0' w}{\phi_0}
\end{aligned} \tag{24}
$$

Performing a change of variable defined by $w = r f(r)$ than from (24) we obtain for f

$$\varepsilon^2 f'' + f' \left(\frac{4\varepsilon^2}{r} - \gamma g_0 \right) + f \left[\sigma^2 + g_0 \left(\frac{4 - 3\gamma}{r} + \frac{\phi'_0}{\phi_0} \right) \right] + \frac{g_0}{r\, \phi_0} \int \frac{8\pi\; r\; f\; \rho_0}{3 + 2\omega} dr = 0,$$

$$(25)$$

the pulsation equation transforms to the following Eddington equation in the limit $\omega \to \infty$ and $\phi = const$:

$$\varepsilon^2 f'' + f' \left(\frac{4\varepsilon^2}{r} - \gamma g_0 \right) + f \left[\sigma^2 + (4 - 3\gamma) \frac{g_0}{r} \right] = 0. \qquad (26)$$

Let's put as a first approximation f equal to a constant then from (25)

$$\sigma^2 = g_0 \left(\frac{4 - 3\gamma}{r} + \frac{8\pi\; \int r\; \rho_0 dr}{r\,(3 + 2\omega)\, \phi_0} + \frac{\phi'_0}{\phi_0} \right).$$

The stability condition one can obtain when put $\sigma^2 = 0$, in this case critical value for γ is

$$\gamma_{cr} = \frac{4}{3} + \frac{8\pi\; \int r\; \rho_0 dr}{3\,(3 + 2\omega)\, \phi_0} + \frac{r\; \phi'_0}{3\phi_0}. \qquad (27)$$

To begin to understand this problem we have model of gas density distribution for clusters of galaxies [5], with functions of a matter distributions for spherically symmetric objects:

$$\rho_0 = \rho_c \left(1 + \left(\frac{r}{R} \right)^2 \right)^{-1.5\beta}, \qquad (28)$$

where the core radius R, central density ρ_c , and the number are parameters varies from cluster to cluster but has typical value the order of 2/3 which implies that the gas mass generally increases linearly with radius. There are analogous form of density distribution for globular star cluster [13], in this case $\beta = 1$. Comparing this relation with a numerical solution of Lane-Emden equation for an isothermal sphere [8] one can say that for particular case the relative error is less than 5 %. Although these models are simplistic, it exhibits many of the key features of the more complex problem. Using

relations describing the clusters of galaxies like (28), it is possible to give a qualitative answer to the question that reaches critical values of γ.

Thus, under the assumption of matter distributions (28) we have find from (15) expression for not perturbed value of scalar field ϕ_0

$$\phi_0 = \int \frac{-8\pi \int \rho_c (1 + \frac{r^2}{R^2})^{-\frac{3\beta}{2}} r^2 dr + (3 + 2\omega)C_1}{(3 + 2\omega)r^2} dr + C_2, \qquad (29)$$

In the previous expressions one can exlain the ϕ_0 using the Gauss hypergeometric function.

For large radius where the effect of the central conditions is very weak the solution should asymptotycally approach the exterior solution. It is know that the scalar field is a constant outside the matter distributions [14]. In empty space there is usual Newtonian universe, but inside "gravitation constant" depend on matter distributions. Thus, the integration constants C_1 and C_2 are determined by matching the interior solution (29) to the usual exterior Newtonian vacuum solution $\phi_0 = G$, $\phi_0' = 0$.

To construct a spherically symmetric configurations for given matter distributions (28), we choose $\beta = 1$ and $\beta = 3/2$. Knowing ϕ_0 and ρ_0 allows for the determination of the critical values of γ for linear adiabatic radial perturbations of spherically symmetric gas spheres, using equation (27). The whole structure of the perturbated gas spheres is thus determined.

The region of classic Newtonian stability is the rectangle $0 < n < 3$. However, the conjecture that $\gamma > 4/3$ is a necessary and sufficient condition for stability in the Newtonian approximation of Jordan-Brans-Dicke theory of gravitation is shown to be false. The critical values of γ depends on the values of the Brans-Dicke parameter ω and on the values of the central dencity ρ_c. For both casses here $\beta = 1$ and $\beta = 3/2$ for negative ω is seen γ to be less than $4/3$ in the instability region. Since scalar field effects are stabilizing, one expects stability in wider area than of the Newtonian region.

4 Conclusion

In the present work, we have thoroughly analyzed static spherically symmetric configurations in the framework of the Newtonian approximation of Jordan-Brans-Dicke theory of gravitation. The classic result for adiabatic

radial pulsations of Newtonian gas spheres is given by the marginal stability $\gamma > \frac{4}{3}$. Since scalar field effects are stabilizing, one expects stability in extended area of the Newtonian region. The behavior of the stable equilibrium solutions for the spherically symmetric configurations considered here, it emerges that the more compact a model is, the more stable it is. Furthermore, in the Jordan, Brans-Dicke theory exist the stable configurations for any polytropic index. The similar effect in classic Newtonian theories does not exist. Whichever it might be, it is very likely that the same phenomena could also occur for clusters of galaxies, such as neutron stars and white dwarfs. In this sense, the results obtained in this paper can be regarded as of a general nature.

The goal of the present work was to present a stability analyze for model of gas density distribution. This configurations can be used to compare with a numerical stability code for configurations with polytropic equations of state. Together with this, a number of new physical features have been displayed concerning the radius-mass relation.

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